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RESEARCH

Correlation of Thermo-Magnetic Properties of Non-Interacting Two-Electron Parabolic Confined Quantum Dots under External Parameters

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ABSTRACT

The main interest of this study is to investigate the correlation of thermo-magnetic properties with respect to each other on the basis of confinement potential strength, external magnetic field, and temperature dependence. Analytically calculated the bound state energy of the harmonic oscillatory potential using Nikiforov-Uvarov formalism and numerically calculated the characteristic function of the thermodynamic properties partition function, free energy, magnetization and magnetic susceptibility with statistical quantum mechanics extending into the harmonic oscillator potential: Many comprehensive studies from the theoretical point of view were conducted on magneto-thermal properties, but all in all, they did not place emphasis on the functional dependence of the correlation between magneto-thermal properties and their impact on the behavior of a system. We tried to put place a novel approach to investigate the correlation impact of magnetic-thermal quantities dependent on the external magnetic field, confinement potential, and temperature. We divulged comprehensive information about the system to put together this guide for the analysis and interpretation of the interrelation. Taking into consideration free energy as a functional center of magnetic and thermodynamic properties, we calculated and graphically simulated the interrelation of free energy, magnetic susceptibility, and magnetization. The nonlinear correlation between free energy and susceptibility exhibited. Strong external magnetic field and confinement potential strength used to determine an optimized free energy to its critical value. At sufficiently low magnetic field and confinement potential strength determines the minimum value of free energy. The maximum value of magnetic susceptibility exhibited in certain intermediate confinement and magnetic field. As confinement increases, magnetization linearly decreases. In cases of sufficiently high confinement potential and temperature, the shortest curve of magnetic response is displayed.

KEYWORDS

Magneto-thermal, Confinement potential, Magnetic susceptibility, magnetization, Quantum dot, non-interacting, Nikiforov-Uvarov, Schrödinger equations.

INTRODUCTION

In condensed matter physics, quantum dots have recently become a very interesting topic [**1-3**]. There are two causes for it. First and foremost, quantum dots have considerable promise for use in single-electron transistors, quantum computers, solar cells, and quantum dot lasers, among other microelectronic devices. Secondly, and perhaps even more importantly, quantum dots can be thought of as a little laboratory where all of the predictions of quantum mechanics can be thoroughly examined. Any theory including quantum dots must take into account the nature of the confining potential.

In situations involving a symmetric confinement potential the rich spectrum of crossing energy levels as a function of magnetic field and strong interaction effects 4 is now fairly well defined 5 recently taking into account the characteristics of quantum dots under less symmetric configuration potentials has become the focus of attention the far-infrared excitation spectrum of a completely parabolic quantum for instance is rather simple and widely known today but the decreased symmetry adds additional aspects to the spectrum whose interpretation is significantly more difficult to understand [**6**]. Different models of confinement potential (CP) are easier to apply than analytical mathematical formulas in the realm of the physical world that are used to force a particle system to act within a specific region of space. It has been employed to model several physical phenomena. This is because it provides us with information on time as well as a relatively less expensive approach to simulating physical systems of interest compared to experimental and advanced computational approaches [**7**]. In the last few years a lot of research has been done on the heat capacity and entropy of quantum dots and other comparable low-dimensional systems in the presence of a magnetic field [**8-9**] nevertheless the confining potential in the majority of these studies has been determined using either a parabolic potential model (PPM) or a square-well model (SQM) when a single magnetic ion and a perpendicular magnetic field are present Nguyen and Peeters have examined thermodynamic quantities like heat capacity magnetization and susceptibility [**10**] however it is extremely difficult to interpret a many-electron quantum dot when electron-electron correlations are taken into consideration. In recent years, there has been a lot of experimental and theoretical research on low-dimensional structures, and quantum dots in particular [**11,12**]. Discrete energy levels with a few meV spacing are produced when charge carriers are confined in quantum dots.

The physics of low-dimensional semiconductor materials is significantly influenced by the quantum dot confinement potential forecast. The morphological structures and properties of WBSs can be governed by their material design, dimensionality engineering, and device engineering, in addition to attempting to make sense of the interplay between material growth, device structure, and application scenarios [**13**]. It should be noted that theoretical investigations of the physical characteristics of quantum dots depend on an understanding of the theoretical confinement potential profile.

Theoretically, understanding the concept of the confining potential which is frequently shown as Gaussian confinement, a spherical harmonic, a pyramidal potential, a ring-shaped oscillator, or a double ring-shaped oscillator is crucial to studying quantum dots. The electrical and optical characteristics of quantum dots have been thoroughly studied over the last few decades in a variety of external conditions, including temperature, impurity, pressure, magnetic field, electric field, electron-phonon occurrence, and spin-orbit correlation [**14–16**].

The thermal and magnetic characteristics of electrons confined in quantum dots in the presence of an external magnetic field have been the subject of extensive research in recent years [**17**]. The effect of mean energy, heat capacity, entropy, magnetization, and susceptibility on temperature and magnetic field has been theoretically worked out by the authors [**18–20**]. The thermodynamic features of nanodimensions are a fascinating window for the newly emerging study of physics [**21**]. Theoretical investigation is done on the thermodynamic behaviors of the double-ring-shaped QDs [**23**], such as energy, entropy, heat capacity, and magnetic susceptibility [**22–23**].

A potential indicator of the influence of phonons on the relaxation of spin between triplet and singlet states in a quantum dot containing two electrons investigated [**24**]. An external magnetic field provided perpendicular to the plane will boost the rotational frequency's spin-orbit impact [**25**] Assist in optimizing a system's free energy so that there is no longer any system disorder.

By assuming a solution-free particle motion with an account of the SE along the axial direction, almost all authors discovered an exact solution for the canonical partition function [**26**]. As a result, calculations were made about the system's magneto-thermal behavior. As such, many authors have intensively focused on the dependence of external fields merely without considering the interrelated thermo-magnetic properties. In this study, we investigated the correlation between thermomagnetic properties such as magnetization and magnetic susceptibility evoked due to its exposure to external field effects like temperature and external magnetic field.

The magnetization and susceptibility of a two-electron parabolic quantum dot are studied in the presence of electron–electron and spin–orbit interactions as a function of the magnetic field, and temperature is studied [**27**]. Considering the canonical partition function to uncover known as the von Neumann entropy that is fundamental to probing the thermodynamic information [**28-29**] of a system without loss of generality of laws of thermodynamics and thus the free energy obtained from the partition function with having regard to we approached to calculate the correlation of free energy which, shows the tendency in resisting no more system disorder as a function of magnetization, as a function of magnetic susceptibility, and susceptibility as a function of magnetization dependence on the external magnetic field, confinement potential, and temperature is investigated.

MATHEMATICAL FORMALISM

The Nikiforov-Uvarov (NU)

Based on the solution of hypergeometric second-order differential equations using unique orthogonal functions, the Nikiforov-Uvarov (NU) technique was developed [**30**].

When a potential is specified, the Schrödinger or Schrödinger-like equations in spherical coordinates can be solved systematically to get the precise or specific solutions by reducing them to a generalized equation of hyper-geometric type with the required coordinate transformation r→s.

The Schrödinger equation's solution may be applied more directly, simply, and elegantly with the help of the NU approach. The discrete spectrum's energy levels are produced from the particle system by applying the NU approach to solve the eigenvalue equations following the separation of the SE. The primary formula is strongly related to the procedure [**31**]. When a physical system oscillates around a mean value at one or more distinguishable frequencies, it is said to be in harmonic motion [**32**]. Such a system describes a bounded particle traveling in a potential well as having a velocity that increases quadratically with the distance from the minimum of the potential well.

$$\frac{d^2U(r)}{dr^2}+\frac{2\mu}{\hbar^2}\left[E-\frac{1}{2}\mu\omega^2r^2-\frac{\hbar^2\ell(\ell+1)}{2\mu r^2}\right]R(r)=0 \quad (1)$$

It would be helpful to incorporate dimensionless variables to make this more mathematically manageable.

$$r=\rho\alpha,\quad \alpha=\sqrt{\frac{\hbar}{\mu\omega}},\quad \epsilon=\frac{E}{\hbar\omega} \quad (2)$$

Equation (3.8) can be expressed

$$\frac{d^2U(\rho)}{d\rho^2}+\frac{2\mu}{\hbar^2}\left[2\epsilon-\frac{\hbar^2\ell(\ell+1)}{\rho^2}-\rho^2\right]U(\rho)=0 \quad (3)$$

By performing transformations $\rho^2 = s$ and $U(\rho)\to\psi(s)$ in equation (3.9), we can obtain an equation by rewriting it in terms of s.

$$\frac{d^2\psi(s)}{ds^2}+\frac{1}{2s}\frac{d\psi(s)}{ds}+\frac{-s^2+\beta s-\ell(\ell+1)}{4s^2}\psi(s)=0 \quad (4)$$

In cases when the range of the variable s is $0\le s\le\infty$. additionally, we employed the definition and derivative, respectively.

$$\frac{d^2U(\rho)}{d\rho^2}=4s\frac{d^2\psi(s)}{ds^2}+\frac{1}{2s}\frac{d\psi(s)}{ds} \quad (5)$$

$$\beta^2=2\epsilon \quad (6)$$

To determine the pertinent polynomials, do the following

$$\tilde{\tau}=1,\ \sigma(s)=2s,\qquad \tilde{\sigma}=-s^2+\beta s^2-\ell(\ell+1) \quad (7)$$

Putting the polynomials in the provided equation (3.13) gives the polynomial $\pi(s)$

$$\pi(s)=\frac{1}{2}\pm\sqrt{s^2+(2k-\beta^2)s+\ell(\ell+1)} \quad (8)$$

Setting $\Delta=b^2-4ac=0$ will solve the quadratic form problem beneath the square root sign of equation (3.14). This quadratic's discriminant equals zero.

$$\Delta=4k^2+\beta^4-2k\beta^2-4(\ell(\ell+1))=0 \quad (9)$$

$$k^2-k\beta^2+\frac{\beta^2}{4}-4\left(\ell(\ell+1)+\frac{1}{4}\right)=0 \quad (10)$$

$$k_\pm=\frac{\beta^2\pm\sqrt{1+4\ell(\ell+1)}}{2} \quad (11)$$

Equation (3.14) may be solved by substituting the two values of k from equation (3.16), yielding the four alternative forms of π(s).

$$\pi(s)=\begin{cases} s+\frac{\sqrt{1+4\ell(\ell+1)}}{2}, for\ \ k_+=\beta^2+\frac{\sqrt{1+4\ell(\ell+1)}}{2} \\ s-\frac{\sqrt{1+4\ell(\ell+1)}}{2}, for\ k_-=\ \beta^2-\frac{\sqrt{1+4\ell(\ell+1)}}{2}\end{cases} \quad (12)$$

For this value of π(s) in the range (0, ∞), one of the four values of the polynomial π(s) is just appropriate to achieve the bound-state solution a negative derivative [**34**]. Consequently, the best way to represent π(s) is determined to be;

$$\pi(s)=\frac{1}{2}-s+\frac{\sqrt{1+4\ell(\ell+1)}}{2} \quad (2.13)$$

For $k_-=\ \beta^2-\frac{\sqrt{1+4\ell(\ell+1)}}{2}$. By using $\pi(s)$ given in equation (3.13) and remembering $\tilde{\tau}=1$ we can obtain expression $\tau(s)=\tilde{\tau}+2\pi(s)$ to introduce

$$\tau(s)=2+\sqrt{1+4\ell(\ell+1)}-2s \quad (2.14)$$

And the derivative of this expression would be negative, i.e., $\tau'(s)=-2<0$ where $\tau'(s)$ represents derivation of (s). The expression $\lambda = k\_+\pi'(s)$ and $\lambda_n=\frac{-n\tau'(s)-n(n-1)\sigma''}{2}$, gives $\lambda=\beta^2-\frac{\sqrt{1+4\ell(\ell+1)}}{2}-1$, $\lambda_n=2n$. When we compare these expressions, $\lambda=\lambda_n$, one can obtain the energy of the harmonic oscillator,

$$\beta^2-\frac{\sqrt{1+4\ell(\ell+1)}}{2}-1=2n \quad (2.15)$$

$$\frac{E}{\hbar\omega}=2n+\ell+\frac{3}{2} \quad (2.16)$$

For 3D harmonic oscillator

$$E=\hbar\omega_0\left(2n+\ell+\frac{3}{2}\right) \quad (2.17)$$

In case of 2D harmonic oscillator

$$E_{n,\ell}=\hbar\omega_0(2n+\ell+1) \quad (2.18$$

ANALYTICAL SOLUTION OF NON-INTERACTING TWO ELECTRON

Von Neumann entropy, a crucial parameter in quantum information theory, was utilized in this investigation [28] to examine intricate thermodynamic properties and magneto-thermal properties in the presence of a magnetic field running parallel to the z-axis, creating an extra term in the consideration of two electrons locating at the center. The applied external magnetic field plays the role of a chemical potential. The Hamiltonian matrix is given as:

$$H = \begin{pmatrix} \frac{1}{4}\hbar\omega + g^*\mu_B B & 0 & 0 & 0 \\ 0 & \frac{1}{4}\hbar\omega & 0 & 0 \\ 0 & 0 & -\hbar\omega\frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4}\hbar\omega - g^*\mu_B B \end{pmatrix} \quad 3.1$$

In actuality, the partition function encapsulates all information [36] about a physical system and is used to characterize its statistical characteristics at a specific inverse temperature [35,36]. Since its computational precision is essential to any statistical investigation of quantum systems and events [37], all thermodynamic observables may be computed as soon as the energy eigenvalues of the underlying physical system are known. This isn't the case, however, if one is interested in observable behaviors and has access to the system Hamiltonian, which must be diagonalized in order to acquire the z-direction. In this situation of $S_z = 0$, the z-component of the magnetic moment disappears for singlets and triplets, but it does not vanish for $S_z = \pm 1$.

The energy term functionality expressed mathematical as Zeeman Effect $E_s = \pm g^*\mu_B S_z$ leads to the $E_{\pm}^t = \frac{1}{4}\hbar\omega_c \pm g^*\mu_B B_z$. Hence, only the energy splitting occurs for states with parallel spins; the other state is left degenerate.

Utilizing the reduced density matrix method is a tool for determining and probing information about the system through von Neumann entropy [38]. For the von Neumann entropy, begin with the thermal condition (based on its eigenstates) and the occupancy level (per dimer) to simulate the entropy as per dimer of a system. The applicability of the von Neumann entropy has eluded direct analytic continuation and a wider sense regime of relevancy than what might be anticipated from other methods [38]. That is why this study relied on its significance in contributing to providing all the information it intended to probe.

The probability of finding the system in an excited state does not vanish even at zero temperature; the density operator does not reduce to the projection onto the non-degenerate ground state of the system and thus does not describe a pure state with statistical entropy equal to zero. This makes it necessary to calculate the entropy per dimer for the quantum case, also known as the von Neumann entropy. The von Neumann entropy expresses the uncertainty regarding the measurement result. The partition function (per dimer) is as follows when we begin with the thermal state (based on its eigenstates):

$$z\rho = \frac{1}{z}\begin{pmatrix} e^{\beta\hbar\omega} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{-\beta\hbar\omega\boldsymbol{B}} & 0 \\ 0 & 0 & 0 & e^{\beta\hbar\omega\boldsymbol{B}} \end{pmatrix} \quad (3.2)$$

Where

$$\mathrm{z} = \mathrm{tr}(\mathrm{z}\rho) = \mathrm{e}^{\beta\hbar\omega} + 1 + 2\cosh(\beta\hbar\omega\mathbf{B}) \quad (3.3)$$

Where we have introduced an energy offset of $\frac{J}{4}$ the triplet and singlet energies become $E_{offset}^s = -J$ and $E_{offset}^t = 0$). Note that ρ is independent of the offset since we normalize it to $tr(\rho) = 1$.

Three distinct degrees of occupancy of the energy state were seen in the presence of a magnetic field, although the singlet states remained unchanged. The lowest triplet is the only one we take into account. components corresponding to$S_z = 1$. Thus, the spin contribution to the Zeeman shift for a spin-singlet and for a spin-triplet state may be respectively written as

$$H_s = g\mu_B B S_z \quad (3.4)$$

Evaluating free energy is a central endeavor in magneto-thermodynamics [39]. It requires continuous efforts to harvest more thermo-magnetic effects, such as from correlation-dependent phenomena due to exposure to independent fields like external magnetic fields, temperature, potential confinement strength, and pressure.

When utilizing the free energy per dimer for the quantum system

$$F(T,B) = -\frac{1}{\beta}\log\left(e^{\beta\hbar\omega} + 1 + 2\cosh(\beta\hbar\omega\boldsymbol{B})\right) \quad (3.5)$$

For the quantum situation, the magnetic susceptibility χ and its dependency on B at various temperatures are described, and the magnetization m is calculated. In the last 10 years, a lot of research has been done on such systems' spin magnetization [40].

The magnetization's ability to provide details about the multiparticle dynamics of the dots under an external magnetic field is what makes it interesting [41]. Despite the interesting physics involved, studying the magnetic characteristics of quantum dots can provide us with more tools to manipulate electronic magnetism in nanoscale structures.

$$m(T,\boldsymbol{B}) = \left(\frac{\partial F(T,\boldsymbol{B})}{\partial \boldsymbol{B}}\right)_T = \frac{2\hbar\omega\sinh(\beta\hbar\omega\boldsymbol{B})}{e^{\beta\hbar\omega} + 1 + 2\cosh(\beta\hbar\omega\boldsymbol{B})} \quad (3.6)$$

The susceptibility for the quantum case is given by:

$$\chi(T,\boldsymbol{B}) = \frac{\partial m(T,\boldsymbol{B})}{\partial \boldsymbol{B}} = 2\beta\mu_B^2 g^{*2}\frac{2 + (1 + e^{\beta\hbar\omega})cosh(\beta\hbar\omega\boldsymbol{B})}{(e^{\beta\hbar\omega} + 1 + 2\cosh(\beta\hbar\omega\boldsymbol{B}))^2} \quad (3.7)$$

DISCUSSION AND RESULT

Non-interacting Parabolic Confined Electron Quantum Dots.

In LDSN, using the NU method, the energy eigenvalue and the wave functions of an electron confined in a 2D quantum dot are calculated vibrational [**42**]. This study specially considered the NU method to obtain eigenvalues under exposure to external fields and extended it to solve thermodynamic properties employing partition functions in the realm of occupancy states and spin interaction in the realm of magnetic field and confinement strength (optical energy). Additionally, magneto-thermal properties are solved. We used the material parameters of GaAs QD. A few electrons trapped in parabolic confined potential produce a rich variety of physical phenomena in a perpendicular magnetic field [**43**]. The numerical value of material parameters for a GaA is a typical value. $\hbar\omega_0 = 3\,meV$, $\mu_B = \frac{\hbar e}{2\mu} = 0.87\,meV/T$ is the effective Bohr magneton. The orbital degeneracies at ***B*** = 0 are lifted in a magnetic field in the presence of magnetic field the cyclotron energy ($\hbar\omega_c = 1.76\,meV$) for magnetic field (**B**=1 T). Experimental investigation subjected to an external magnetic field and electrostatic confinement potential, the singlet-triplet is altered in relation to the energy splitting and spin relaxation time, which shows a non-monotonic dependence [**44**]. That highly agreed with our theoretical study the correlation of parameters exhibited non-linearity dependence. In comparison an experimental work with that theoretical study it is simply fundamental physical mechanism is captured by a straightforward theoretical model that is developed.

When it comes to information theory or best one important theme that unites theories of optimization is control theory of the sciences from a free-energy approach. As a result, the optimized free energy maintains the expected utility and expected value of incentives. This demonstrates a surprise in both the estimated cost and the forecast error. According to the free energy principle, the amount is optimized, implying that free energy framework work could lead to a unification of global brain theories [**45**]. Our findings demonstrate that the strength of the confinement potential and the external magnetic field both improve disorder manipulation (minimization); as a result, free energy is optimized since the frequency of the system is affected in both scenarios. When exposed to a strong magnetic field and confinement potential, the relationship between magnetic susceptibility and the optimum free energy curve is linear and tends to be shorter.

The results presented here, while not comprehensive, are based on accurate answers due to the parabolic confinement nature of our system, environment, and confinement strength. This facilitates the discovery of significant physical characteristics that are hidden from view when depending just on widely used perturbative or approximative techniques, including Markovian assumptions. Additionally, it makes it possible to rule out incorrect predictions that result from using inaccurate approximations for regimes that are beyond their scope.

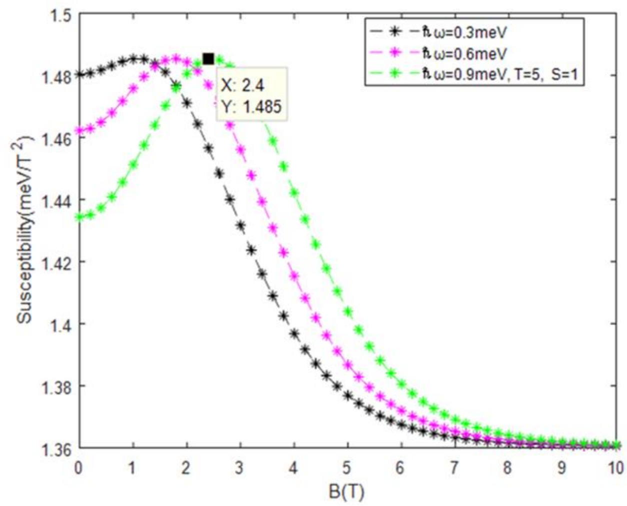

Fig. 1. Susceptibility (meV/T^2) as function of external magnetic fields (Tesla) with various confinement potentials $\hbar\omega = 0.3$ meV, $\hbar\omega = 0.6$ meV and $\hbar\omega = 0.9$ meV, constant temperature T=5 K.

It is demonstrated in **Fig. 1** by the observed phenomenological influence of cyclotron frequency that there is a field-tuned susceptibility between thermally populated excited magnetic states. The spin states involved in the tuning process as well as the variation in confinement potential and strength potential at 5 K have been determined and it shown good agreement with [**46**]. Between the signs of the susceptibility's second derivatives with respect to temperature and field, a consistent relationship is seen.

It is demonstrated in **Fig. 1** by the observed phenomenological influence of cyclotron frequency that there is a field-tuned susceptibility between thermally populated excited magnetic states. The spin states involved in the tuning process as well as the variation in confinement potential and strength potential at 5K have been determined. Between the signs of the susceptibility's second derivatives with respect to temperature and field, a consistent relationship is seen. As depicted in **Fig. 1**, susceptibility increases at sufficiently low external magnetic fields. At zero magnetic field intensity there is static susceptibility, which increases with increment of confinement potential. However, as magnetic field intensity increases the magnitude of susceptibility increases and reaches its peak value and decreases monotonously for a given value of confinement potential value the peak value of susceptibility shift toward high magnetic field intensity.

The simulation result shown in **Fig. 2** is from equations (3.5) and (3.7). In contrast to free energy as a function of optical energy (confinement potential), free energy as a function of susceptibility exhibits a non-linear relationship. For lower confinement potential, free energy possessed the highest value. As potential confinement increases, free energy linearly begins increases slower, but magnetic susceptibility increases quicker until it reaches its maximum value. Meanwhile, both magnetic susceptibility tends to decreases and free energy again linearly increases until they reach their cut-off value with increments of confinement potential. From **Fig. 2**, one can observe that in the case of sufficiently low confinement potential strength the minimum value of free energy and maximum value magnetic susceptibility exhibited. At sufficiently high confinement potential free energy and susceptibility shows independency on magnetic field.

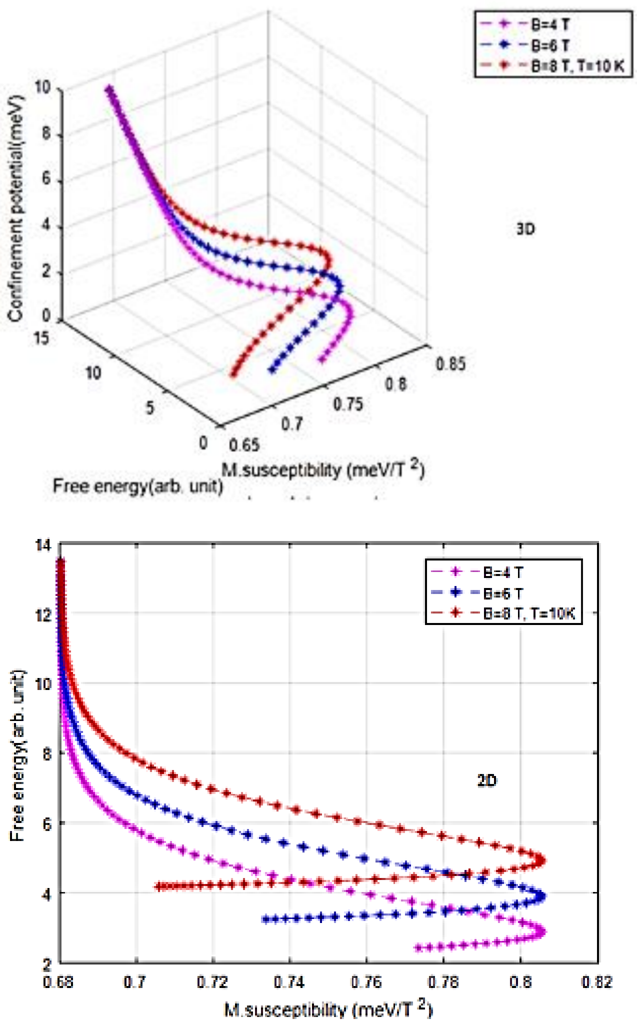

Fig. 2. (a) Illustration in 3D and **(b)** 2D Free energy (arb.unit) versus Susceptibility (meV/T^2) as function of confinement potential ($\hbar\omega = 0 - 10$ meV) for various magnetic field strength (B=4 T, B=6 T, B=8 T) in constant value of temperature (T=10 K).

Table 1. Statistical data show the correlation curve of magnetic susceptibility versus free energy.

Mag. Field	M. Susceptibility (meV/T^2)			Free energy (arb. unit)		
	Mini. value	Max. value	Range	Mini. value	Max. value	Range
4 T	0.6802	0.8052	0.125	2.425	13.47	11.04
6 T	0.6802	0.8052	0.1249	3.249	13.47	10.22
8 T	0.6804	0.8052	0.1248	4.19	13.47	9.278

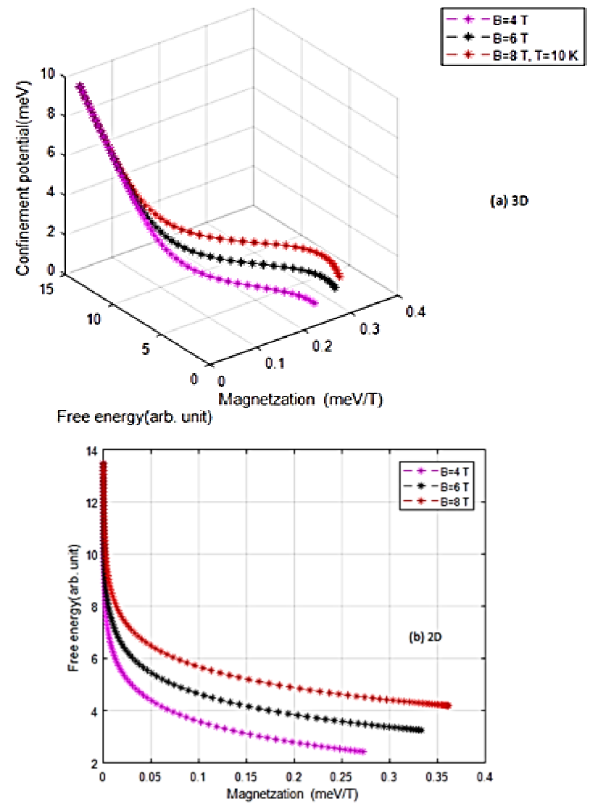

Fig. 3. (a) Illustration 3D and (b) 2D Free energy (arb.unit) versus Magnetization (meV/T) as function of confinement potential ($\hbar\omega = 0 - 10$ meV) for various magnetic field strength (B=4 T, B=6 T, B=8 T) in constant value of temperature (T=10 K).

The result in **Fig. 3(b)** is from simulated solution of combination of equations (3.5) and (3.6), the minimum value free energy is observed at lower confinement potential since the effect of quantum confinement induced strain in quantum dots is significant. However, the magnetization electrons exhibits lower possible value due degrees of freedom not exposed to minimum alignment forces. Meanwhile, as confinement potential increases, free energy increases even magnetization shows a decrement until it reaches its cut-off value as potential confinement increases. From **Fig. 3(b),** one can observe that the variation of magnetic field put impact the at reasonably low confinement potential in both cases for free energy and magnetization. The value for magnetization becomes merged as confinement potential becomes stronger external magnetic fields is more insignificant than for weak confinement potential.

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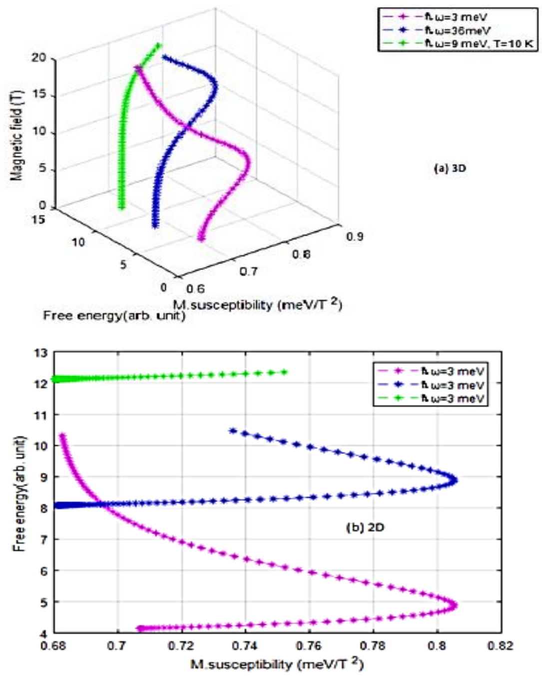

Fig. 4. (a) illustration 3D and (b) 2D, Free energy (arb.unit) as function of Susceptibility (meV/T^2) dependence on external magnetic field ($B = 0 - 20$ T) for various confinement potential strength ($\hbar\omega = 3$ meV, $\hbar\omega = 6$ meV and $\hbar\omega = 9$ meV) in constant value of temperature (T=10 K).

Table 2. Statistical data correlation of magnetic susceptibility between free energy as function magnetic fields with various confinement potentials**.**

Conf. potential	M. susceptibility (meV/T^2)			Free energy (arb. unit)		
	Min. value	Max. value	Range	Min. value	Max. value	Range
3 meV	0.6825	0.8052	0.1227	4.143	10.32	6.173
6 meV	0.6812	0.8052	0.1241	8.084	10.47	2.386
9 meV	0.6802	0.752	0.07173	12.12	12.34	0.2213

Table 2 Coincide with **Fig. 4** the range magnetic susceptibility and free energy curve is broader in case of lower confinement potential. In case susceptibility saturated quickly due to strong confinement before attaining its maximum value. The dominance of magnetic field beside confinement potential optimize for shortest interval.

As depicted in **Fig. 4(b),** the correlation of two dependent variables, free energy and susceptibility, behaved differently based on confinement potential as a function of magnetic fields. As one can observe from Fig. 4.4, for the confinement potential ($\hbar\omega = 3\,meV$) the free energy decreases for an increment of magnetic susceptibility and turned back to its critical minimum value. Moreover, the magnitude of free energy increases as value of confinement potential increases. As the magnetic field increases and magnetic susceptibility reaches critical maximum value, then turned back, and both free energy and susceptibility decrease until they reaches their critical minimum values even if the external magnetic field kept increasing. For the value ($\hbar\omega = 6\ meV$) as **Fig. 4(b)**, shows both free energy and susceptibility linearly increase as the magnetic field intensity increases until magnetic susceptibility reaches its maximum critical value, and then susceptibility begins to decrease, but free energy keeps increasing as magnetic field strength independently increases. In the case of ($\hbar\omega = 9\ meV$), both free energy and magnetic susceptibility show proportional increases for a very small range and reach their cut-off values. The uniqueness of this result is due to the supplementary effects of magnetic field and confinement strength in quantum systems. The win-win dominance of CP and external magnetic field enhances the proportional increment of free energy and magnetic response within a certain limit. On the other hand, the dominance of the magnetic field over CP exhibited two phases of increased magnetic response. That possessed the maximum value with respect to the record of the decrease of free energy in both phases of susceptibility. According to the research, up has the lowest total energy barrier, and the smoothest magnetization reversal was accomplished [**47**].

As **Fig. 5** shows, the strong confinement potential (CP) promotes saturation of magnetization in the shortest range in the correlation of free energy. **Fig. 5** showed free energy as a function of magnetization exhibited in the long-range correlation in the case of lower confinement potential ($\hbar\omega = 3\,meV$) since it allowed the domination of the pinning phenomenon for free energy and magnetization increased. Even after magnetization is saturated, there is free room for free energy to be recorded. A stronger confinement value quantitatively dominates the pinning phenomenon that hinders free energy, and the magnetic domain is also pinned; therefore, magnetization is only responsive in a very short range with respect to free energy.

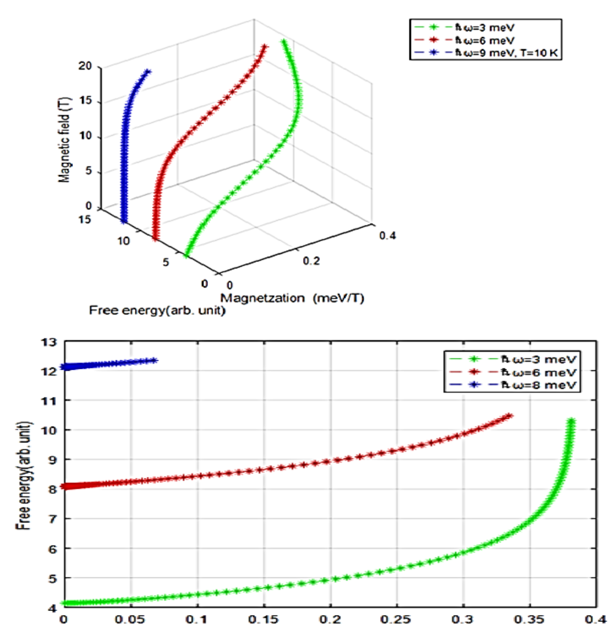

Fig. 5. (a) illustration in 3D and (b) in 2D Free energy (arb.unit) versus Magnetization (meV/T) as function of external magnetic field ($B = 0 - 20\ T$) for various confinement potential strength ($\hbar\omega = 3\,meV, \hbar\omega = 6\ meV\ and\ \hbar\omega = 9\ meV$) in constant value of temperature (T=10 K).

The tendency to resist more disorder in the internal system is measured or calculated through free energy. Therefore, determining the optimal free energy probe over the entire state of the system. Especially the susceptibility of the material to its environment. That is why we place a great emphasis on simulating the free energy of a system under different external parameters as well as its correlation with other material properties.

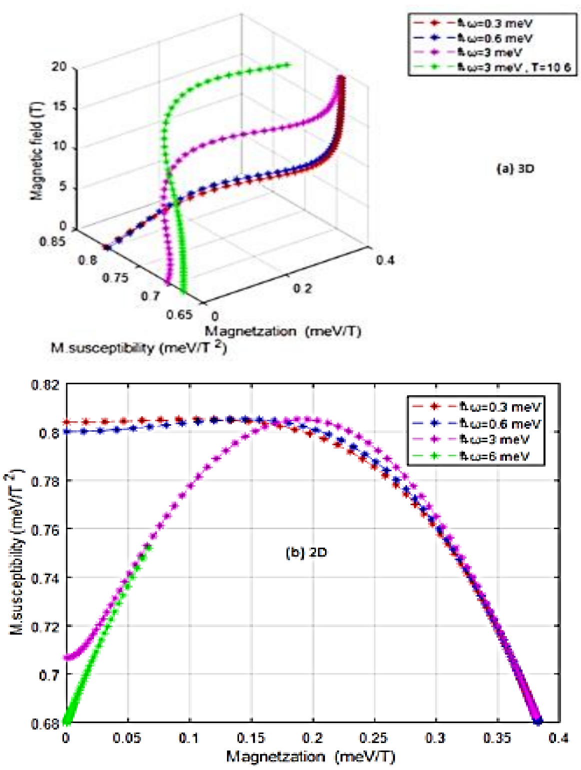

Fig. 6. (a) 3D and (b) 2D: illustration of Susceptibility $(\mathrm{meV/T^2})$ as function of Magnetization (meV/T) dependency on external magnetic field $(\mathrm{B = 0 - 20T})$ for various confinement potential strength $(\hbar\omega = 0.3\ \mathrm{meV}, \hbar\omega = 0.6\ \mathrm{meV}, \hbar\omega = 3\ \mathrm{meV}, \mathrm{and}, \hbar\omega = 6\ \mathrm{meV})$ in constant value of Temperature (T = 10 K).

Table 3. Statistical data the correlation magnetization between magnetic susceptibility as function magnetic field.

Conf. potential (meV)	Magnetization (meV/T)			Magnetic Susceptibility (meV/T^2)		
	Mini. value	Max. value	Range	Mini. value	Max. value	Range
0.3	0	0.3827	0.3827	0.6804	0.8052	0.1248
0.6	0	0.3826	0.3826	0.6804	0.8052	0.1248
3	0	0.381	0.381	0.6825	0.8052	0.1226
6	0	0.3339	0.3339	0.6812	0.8052	0.124

In the account of several magnetization process features, this study is concerned with the high field differential susceptibility (paramagnetic process) type. More thorough ideas on the cause of the magnetization anisotropy were developed as a result of further research **Fig. 6(b)** displays both susceptibility exhibiting the highest value and magnetization linearly increasing in the realm of weak confinement potential at sufficiently low temperatures.

In **Fig. 6(b),** in cases of strong confinement energy, the record shows that both susceptibility and magnetization rose, and susceptibility attained its maximum and turned to decreasing even though magnetization kept increasing. The record of different peaks of susceptibility is observed for different values of confinement energy since QD is very sensitive to confinement potential and has tremendous variations in oscillatory strength and its role in determining magnetic response. From Fig. 4.6, one can observe that temperature affects magnetic susceptibility more than magnetization.

As **Fig. 6(b)** shows, in the asymmetric behavior of magnetic susceptibility and magnetization, the maximum possibility of a magnetic response characterizing susceptibility saturation begins to decrease even though magnetization keeps increasing, thus increasing the intensity of the material's response to an external magnetic field is characterized by spin orientation, while the former is the quickest to reach its maximum value.

An alternate crossover between the spin-split electron levels in the energy spectrum is related to the study of sudden changes in magnetization and susceptibility at low magnetic fields, mostly because of the spin-orbit interaction that has been figured out [**48**]. The extended response curve for magnetic susceptibility and magnetization is shown at the lowest temperature, as seen in **Fig. 7(b)**.

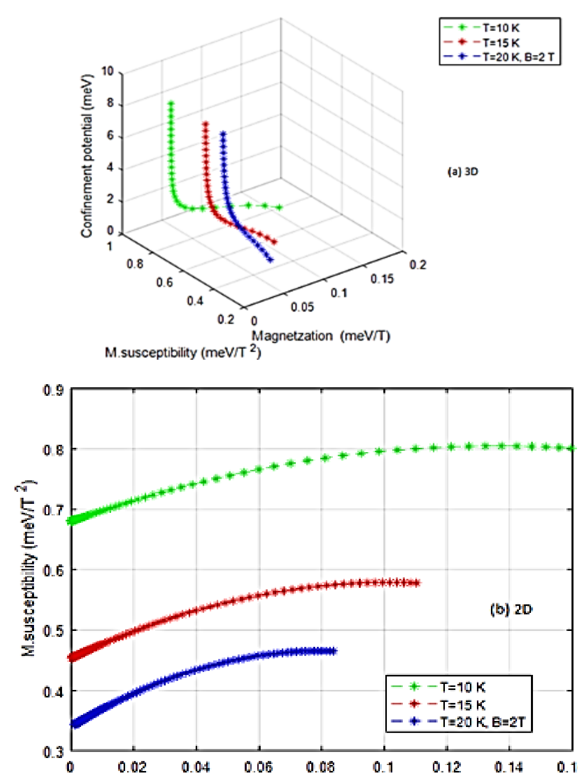

Fig. 7. (a) illustration in 3D and (b) in 2D Susceptibility $(\mathrm{meV/T^2})$ versus Magnetization (meV/T) as function of confinement potential strength $(\hbar\omega = 0 - 10\ \mathrm{meV})$ for various Temperature $(\mathrm{T = 10\ K, T = 15\ K\ and\ T = 20\ K})$ in constant value of magnetic field (B = 2 T).

One can observe from **Fig. 7(b)** the strongest temperature value dominates the responsiveness of their effect of existence for both magnetization and magnetic susceptibility to be quenched. The confinement strength enhanced both susceptibility and magnetization, as shown in **Fig. 7(b)**. The value of susceptibility was recorded in advance in cases of lower temperatures. Similarly, magnetization exhibited a wider range at low temperatures. This signifies the coercively of nanoparticles in the thermal activation of electron moments over the anisotropy barrier in the lower temperature range.

From Table one can observe the range of magnetic susceptibility and magnetization become shorter as value of temperature increases. Thus, thermal agitation put an impact on minimizing magnetic response. Both Magnetic response and magnetization value show highly frequented in case of lower confinement potential. As confinement potential increases both magnetic response and magnetization linearly increase until maximum value of magnetic response reaches its critical maximum value and turned back.

Table 4. Statistical data correlation of magnetization between magnetic susceptibility as function confinement potential with various temperature for (B=2 T).

Temperature	Magnetization			M. susceptibility		
	Mini. value	Max. value	Range	Mini. Value	Max. value	Range
10 K	7.053×10^{-6}	0.1596	0.1596	0.6802	0.8052	0.125
15 K	2.094×10^{-5}	0.1102	0.11	0.454	0.5785	0.1245
20 K	1.05×10^{-3}	0.08366	0.0826	0.3434	0.4651	0.1217

CONCLUSION

The Nikiforov-Uvarov method is utilized to solve the QD Schrödinger equation and calculate Eigen energies' spectra between two electron GaAs quantum dots with harmonic parabolic potential confinement, considering independent variables: the magnetic field strength (B), confining potential ($\hbar\omega$), and temperature (T). In this work, we have shown the interrelation of free energy, magnetization, and susceptibility with the dependence of the external magnetic field, confinement potential, and temperature. Considering the possible statistical occupancy of the energy level, this accounts for the presence of the spin (S_z) and possible degeneracy. We have studied the interdependence of free energy, magnetic susceptibility, and magnetization on the external magnetic field, temperature, and confining potential of the parabolic oscillatory confinement of two non-interacting quantum dots. We investigated the maximum possible peak and the minimum value at which the cut-off magnetic response of the system was indicated. We investigated the interrelated dependent variables experience and their determining effects on magneto-thermal properties in a limited interval.

From a theoretical point of view, many studies investigated the dependence on external parameters without considering the interdependence of thermo-statistical and magneto-thermal behaviors. In contrast, comprehensive information about the system is needed to put together this guide to the analysis and interpretation of the interrelation, taking into consideration free energy as a functional bridge of magnetic and statistical thermodynamic properties at the same pace. This study investigate the interrelationship between intrinsic properties depending the external parameters. Since the confinement length, magnetic field, and temperature are shown to directly rely on these thermal and magnetic variables, the system's parameters can be adjusted to suit a variety of applications. Thus, in the era of scientific technology, information is required to be stored and probed from a scientific point of view, and the systems cannot be found in an isolated external parameters. The significance of our study is pronounced; further investigation will be required due to the higher variability of material parameters to be instrumentalized as a gauge to probe the internal properties of the system.

The discontinuity of magnetic response shows the physical boundary conditions requiring great attention to be investigated for different nanostructure materials. It requires intensive collaboration between chemists, physicists, and materials scientists to better understand the relationship between magnetic characteristics and nanostructure by investigating both the fundamentals and possible applications [**49**]

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CONFLICT OF INTEREST

The authors have no conflicts of interest to declare that are relevant to the content of this article.

DATA AVAILABILITY STATEMENT

Data sets generated during the current study are available from the corresponding author on reasonable request.

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