The Perspective of a Homogenization Approach for Effective Local and Non-local Response of the Elastic Wave Properties of Phononic Metamaterials

J. Flores-Méndez[1,2], R. Pérez Cuapio[2], C. Bueno Avendaño[3], Martín Hernández-Ordoñez[4], M. Aparicio Razo[2], F. Candia García[2], I. Vivaldo-De la Cruz[2], R. C. Ambrosio Lázaro[2], B. Zenteno-Mateo[2,*]

[1]Tecnológico Nacional de México/Instituto Tecnológico de Puebla, Av. Tecnológico, No. 420, Corredor Industrial la Ciénega, Puebla, Pue., CP 72220, México
[2]Facultad de Ingeniería y Electrónica - Benemérita Universidad Autónoma de Puebla, Blvd. Valsequillo y Av. San Claudio, s/n, edif. ING-2, Col. San Manuel, Ciudad Universitaria, Puebla, Puebla, 72570, México
[3]Tecnológico Nacional de México/I.T. Apizaco. Carretera Apizaco-Tzompantepec, esquina con Av. Instituto Tecnológico S/N, Conurbado Apizaco-Tzompantepec, Tlaxcala, CP 90300, México
[4]Tecnológico Nacional de México/ Instituto Tecnológico de Veracruz, Calz. Miguel Ángel de Quevedo 2779, Col. Formando Hogar, Veracruz, Ver. CP 91897, México

*Corresponding author: E-mail: benito.zenteno@correo.buap.mx; Tel: (+52 1) 222 845 2193

DOI: 10.5185/amlett.2020.121581

This review summarizes progress about a recent homogenization theory based on the Fourier formalism for solid phononic crystals, which is valid for arbitrary Bravais lattice and any form of inclusions in the unit cell. The theory provides explicit expressions for the tensors of the effective nonlocal elastic response (dependence on frequency and wave vector), namely the effective dynamic mass-density and compliance matrices. With the use of this theory, our predictions in the quasistatic limit for one and two-dimensional phononic crystals coincide with those of finite-element and asymptotic-homogenization methods. It is also shown that the derived expressions can be applied to phononic crystals with liquid components (two-dimensional sonic crystals) and agree with predictions of the multiple scattering theory. The formalism of non-local effects is fully demonstrated only for a one-dimensional elastic metamaterial having simultaneously negative effective dynamic mass density and elastic shear modulus. The development and applications of this homogenization theory, unlike other formalisms, arises from the inspiration of intense research efforts to simultaneously describe local and non-local effective properties in elastic periodic structures.

Introduction

The concept of metamaterial was initially introduced to explain the striking physical properties of photonic crystals, composed of resonant elements or having a very large dielectric contrast, in a simple and clear manner [**1-6**].

Since Victor Veselago, in 1968, considered for the first time media with simultaneously ϵ and μ negative from a theoretical point of view [**7**]. His pioneering work made it possible to predict, for example, that the phase velocity and energy flow in such media could point in opposite directions. Thus, the media could be considered to have a negative refractive index (n). He also systematically investigated many effects resulting from his findings, including negative refraction at an interface, negative Doppler shifts, etc. As well, he considered the behavior of concave and convex lenses manufactured with such media showing also that a flat slab of material with $n = -1$ could image a point source located on one side of the slab onto two other points, one inside the slab and one on the other side of it (provided that the thickness of the slab was thin enough). Therefore, their realization took the path of engineered structures that have been called, metamaterials, which owes its origin to R. M. Walser who defined them as "Macroscopic composites having a manmade, three-dimensional, periodic cellular architecture designed to produce an optimized combination, not available in nature, of two or more responses to specific excitation" in 1999 [**8,9**].

In 1996 Sir John Pendry, from Imperial College London, presented a practical way to implement an electromagnetic metamaterial, obtaining with the help of his collaborators an artificial material (metamaterial) with negative dielectric permittivity ϵ [**1**].

The next challenge was to obtain metamaterials with negative magnetic permeability, even though in nature there are media with negative permittivity called ferroelectrics, the problem was to manufacture media with negative permittivity artificially. Thus, in 1999 again Pendry and his collaborators devised a way to obtain a medium with negative permeability from "C" shaped resonators [**4**]. A year later, David Smith and his collaborators of the University of California in San Diego were the first to implement in a practical way Pendry's ideas and manufactured a metamaterial that presents simultaneously both negative parameters [**5**].

Subsequently a metamaterial with a negative index of refraction could be experimentally verified [**10**] and such interesting phenomena as super resolution [**6**] and invisibility cloak were demonstrated [**11**].

A photonic metamaterial represents a homogeneous medium with effective electromagnetic response that, in general, turns out to be nonlocal [**12,13**]. The metamaterial concept has been extended to phononic crystals being described as homogeneous media with effective dynamic parameters. Conventional homogenization theories of phononic crystals describe their vibrational properties in the long wavelength limit, i.e. when the microscopic acoustic field smoothly varies inside each component (matrix and inclusion) in the unit cell. However, if the heterogeneous system contains resonant elements or the contrast between the elastic moduli of the components is sufficiently high, the conventional description may fail. For example, a FCC arrangement of soft-rubber spheres in water, having Mie resonances at low frequency, behaves as a double-negative acoustic medium with simultaneously negative effective bulk modulus and mass density [**14**]. In that work, the standard homogenization [**15**] was modified, by using the coherent potential approximation method, to explain the resonant frequency dependence of the acoustic metamaterial parameters. Another approach, namely the multiple scattering theory has had to be applied for calculating effective sound velocity and density of circular-shaped clusters consisting of two-dimensional distributions of rigid cylinders in air in the low-frequency limit [**16,17,18**]. In Ref. [**19**], the method for retrieving effective properties of electromagnetic materials from experimentally-measured reflection and transmission coefficients [**20,21**] has been extended to acoustic metamaterials.

A homogenized phononic crystal, composed of a solid host matrix, is denoted elastic metamaterial [**22**]. Comparing with an acoustic metamaterial in the isotropic case, the elastic metamaterial is characterized by the effective shear modulus, besides the bulk modulus and mass density. Recently [**23**,**24**], applying the effective medium theory for certain elastic metamaterials in two dimensions, which is valid beyond the quasistatic limit, various resonant elastic metamaterials, possessing negative shear modulus and negative mass density, have been proposed.

More recently, more and more attention has been paid to investigating the elastic wave properties of three-dimensional periodic solid-solid (or solid-fluid) media. Within the framework of the homogenization approach in Ref. [**25**], the anisotropy of the effective mass density in the low-frequency limit for homogenized three-dimensional phononic crystals, having solid and liquid materials in the unit cell, was studied. There, the form-factor division approach, which has been successfully applied to calculate effective parameters for photonic metamaterials [**12,13,26**], was employed to reduce the computing time.

A different but not less general homogenization scheme was developed in the work [**27**] where expressions for the fully dynamic effective material parameters, governing the spatially averaged fields by using the plane wave expansion method, were obtained. As it is shown there, the effective material parameters can be calculated for arbitrary frequency and wave number combinations, including but not restricted to Bloch wave branches for wave propagation in the periodic medium.

In the present work, we present a homogenization theory that has contributed to the recent development in the area of acoustic metamaterials to calculate the effective elastic response for phononic crystals of arbitrary Bravais lattice and any type of the inclusions inside the unit cell. The theory is based on the Fourier formalism, but unlike the previous work [**28-31**], it provides the dependences of all the components of the effective mass-density and compliance tensors upon frequency and wave vector. Here, we numerically model metamaterials in one and two dimensions and summarize the main achievements and contributions of our theory to date [**29,32,33**].

Calculation of effective parameters

Let us consider an elastic phononic crystal (PC) characterized by position-dependent mass density $\rho(\vec{r})$ and stiffness tensor $C_{ijkl}(\vec{r})$. In the PC, the displacement vector $\vec{u} = (u_1, u_2, u_3)$ and the Cauchy stress tensor σ_{ij} obey Newton's and Hooke's laws:

$$-\omega^2 \rho(\vec{r}) u_i = \nabla_j \sigma_{ij}, \tag{1}$$

$$\sigma_{ij} = C_{ijkl}(\vec{r}) \nabla_k u_l. \tag{2}$$

Using Voigt notation, we shall rewrite Eqs. (1) and (2) in matrix form:

$$\begin{bmatrix} 0_3 & \nabla_{3\times 6} \\ (\nabla_{3\times 6})^T & 0_6 \end{bmatrix} \vec{v}(\vec{r}) = \bar{\bar{\Omega}}\, \bar{\bar{A}}(\vec{r}) \vec{v}(\vec{r}). \tag{3}$$

Here, we have introduced the nine-dimensional column vector $\vec{v}(\vec{r})$, formed by the components of the displacement and stress vectors as

$$[\vec{v}(\vec{r})]^T = (u_1, u_2, u_3, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6). \tag{4}$$

The 3×6 matrix $\nabla_{3\times 6}$ is defined as

$$\nabla_{3\times 6} = \begin{pmatrix} \nabla_1 & 0 & 0 & 0 & \nabla_3 & \nabla_2 \\ 0 & \nabla_2 & 0 & \nabla_3 & 0 & \nabla_1 \\ 0 & 0 & \nabla_3 & \nabla_2 & \nabla_1 & 0 \end{pmatrix}, \tag{5}$$

whereas the 9×9 matrix $\bar{\bar{\Omega}}$ in Eq. (3) is given by

$$\bar{\bar{\Omega}} \equiv \begin{bmatrix} -\omega^2 I_3 & 0_{3\times 6} \\ 0_{6\times 3} & I_6 \end{bmatrix}, \tag{6}$$

where I_3 (I_6) and 0_3 (0_6) are unity and zero matrices of order 3 (6), and $0_{3\times 6}$ ($0_{6\times 3}$) stands for a 3×6 (6×3) zero matrix. The 9×9 matrix $\bar{\bar{A}}(\vec{r})$ in Eq. (3) is defined in terms of the mass density $\rho(\vec{r})$ and 6×6 compliance tensor $\bar{S}(\vec{r})$ as

$$\bar{\bar{A}}(\vec{r}) = \begin{bmatrix} \rho(\vec{r}) I_3 & 0_{3\times 6} \\ 0_{6\times 3} & \bar{S}_{6\times 6}(\vec{r}) \end{bmatrix}. \tag{7}$$

Because of the periodicity of $\rho(\vec{r})$ and $\bar{S}(\vec{r})$, we can expand the matrix $\bar{\bar{A}}(\vec{r})$ (7) into Fourier series:

$$\bar{\bar{A}}(\vec{r}) = \sum_{\vec{G}} \bar{\bar{A}}(\vec{G}) e^{i\vec{G}\cdot\vec{r}}, \tag{8}$$

where the summation ranges over all the vectors of the reciprocal lattice of the PC. Besides, the nine-dimensional vector $\vec{v}(\vec{r})$ (4) should fulfill the Bloch theorem. Then,

$$\vec{v}(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} \vec{v}(\vec{G} = 0) + e^{i\vec{k}\cdot\vec{r}} \sum_{\vec{G}\neq 0} \vec{v}(\vec{G}) e^{i\vec{G}\cdot\vec{r}}. \tag{9}$$

Since the Bloch wave vector $\vec{k}$ can be considered inside the first Brillouin zone (BZ), we define the average *v*-field as the term with $\vec{G} = 0$ in Eq. (9):

$$\vec{V}(\vec{r}) \equiv e^{i\vec{k}\cdot\vec{r}}\vec{v}(\vec{G}=0) = e^{i\vec{k}\cdot\vec{r}}\vec{V}_0. \quad (10)$$

This averaging procedure corresponds to a truncation of the plane wave expansion (9) so that the Fourier components outside the first BZ are eliminated [**34**]. As follows from Eqs. (3) and (9), the coefficients $\vec{V}_0 = \vec{v}(\vec{G} = 0)$ and $\vec{v}(\vec{G} \neq 0)$ satisfy the algebraic system of equations:

$$\sum_{\vec{G}'} \bar{\bar{D}}\,(\vec{k};\vec{G},\vec{G}')\cdot\vec{v}(\vec{G}') = 0, \quad (11)$$

where

$$\bar{\bar{D}}(\vec{k};\vec{G},\vec{G}') = -\begin{bmatrix} 0_3 & K_{3x6}(\vec{k}+\vec{G}) \\ \left(K_{3\times6}(\vec{k}+\vec{G})\right)^T & 0_6 \end{bmatrix}\delta_{\vec{G},\vec{G}'} - i\bar{\bar{\Omega}}\,\bar{\bar{A}}(\vec{G}-\vec{G}'), \quad (12)$$

$\delta_{\vec{G},\vec{G}'}$ stands for the Kronecker delta, and the 3×6 matrix $K_{3\times6}(\vec{k})$ has the form

$$K_{3\times6}(\vec{k}) = \begin{pmatrix} \vec{k}_1 & 0 & 0 & 0 & \vec{k}_3 & \vec{k}_2 \\ 0 & \vec{k}_2 & 0 & \vec{k}_3 & 0 & \vec{k}_1 \\ 0 & 0 & \vec{k}_3 & \vec{k}_2 & 0 & 0 \end{pmatrix}. \quad (13)$$

We can write the coefficients $\vec{v}(\vec{G} \neq 0)$ in terms of $\vec{V}_0$ employing the equations for $\vec{G} \neq 0$ in (11). We get

$$\vec{V}(\vec{G}) = -\sum_{\vec{G}'\neq0} \bar{\bar{D}}_S^{-1}\,(\vec{k};\vec{G},\vec{G}')\cdot\bar{\bar{D}}(\vec{k};\vec{G}',0)\cdot\vec{V}_0. \quad (14)$$

Here, $\bar{\bar{D}}_S(\vec{k};\vec{G},\vec{G}')$ is a submatrix, obtained from $\bar{\bar{D}}(\vec{k};\vec{G},\vec{G}')$ (12) after eliminating its block rows (columns) with $\vec{G} = 0$ $(\vec{G}' \neq 0)$. Substituting the expression for $\vec{v}(\vec{G} \neq 0)$ (14) and using the relation [**7**] between the inverse of a submatrix $\left(\bar{\bar{D}}_S^{-1}\right)$ and the inverse of the original matrix $\left(\bar{\bar{D}}^{-1}\right)$ into Eq. (11) with $\vec{G} = 0$, we have derived an equation for the amplitude $\vec{V}_0$ of the average *v*-field:

$$\begin{bmatrix} 0_3 & K_{3x6}(\vec{k}) \\ \left(K_{3\times6}(\vec{k})\right)^T & 0_6 \end{bmatrix}\vec{V}_0 = -i\bar{\bar{\Omega}}\,\bar{\bar{A}}_{\text{eff}}(\vec{k})\vec{V}_0\,, \quad (15)$$

where the effective nonlocal-response matrix $\bar{\bar{A}}_{eff}(\vec{k})$ is explicitly given by

$$\bar{\bar{A}}_{\text{eff}}(\vec{k}) = i\bar{\bar{\Omega}}^{-1}\{\bar{\bar{D}}^{-1}(\vec{k};0,0)\}^{-1} + i\bar{\bar{\Omega}}^{-1}\begin{bmatrix} 0_3 & K_{3x6}(\vec{k}) \\ \left(K_{3\times6}(\vec{k})\right)^T & 0_6 \end{bmatrix}. \quad (16)$$

Here, $\bar{\bar{D}}^{-1}(\vec{k};0,0)$ is a 9×9 block, obtained from the inverse $\bar{\bar{D}}^{-1}(\vec{k};\vec{G},\vec{G}')$ of the infinite-size matrix (12), and $\{\ldots\}^{-1}$ symbolizes the inverse of the 9×9 matrix block.

In order to show the usefulness of our main result (formula (16)), we have calculated the effective parameters of various phononic crystals possessing inversion symmetry (**Fig. 1**). The block structure of the matrix $\bar{\bar{A}}_{eff}(\vec{k})$ for such systems has the form

$$\bar{\bar{A}}_{\text{eff}}(\vec{k}) = \begin{bmatrix} \bar{\rho}_{\text{eff}}(\vec{k}) & 0_{3\times6}(\vec{k}) \\ 0_{6\times3}(\vec{k}) & \bar{S}_{\text{eff}}(\vec{k}) \end{bmatrix}, \quad (17)$$

where $\bar{\rho}_{\text{eff}}(\vec{k})$ and $\bar{S}_{\text{eff}}(\vec{k})$ are effective mass-density and compliance tensors.

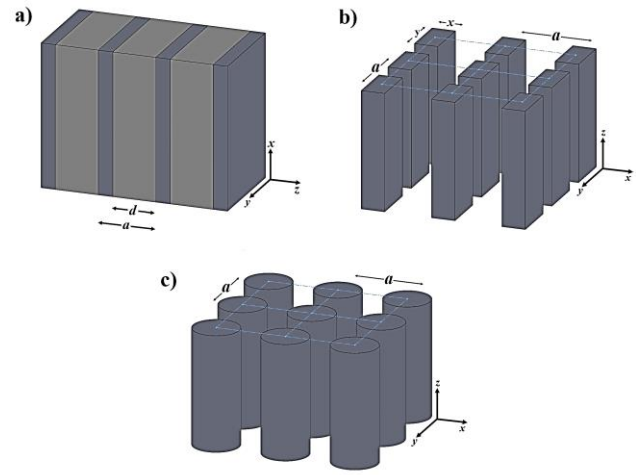


Fig. 1. (a) 1D phononic crystal; (b) 2D phononic crystal and (c) 2D sonic crystal (solid medium embedded in fluid). In all cases *a* is the lattice constant.

Results

Fig. 2 exhibits graphs of the nonzero elements in the matrices $\bar{\rho}_{\text{eff}}$ and $\bar{C}_{\text{eff}} = \bar{S}_{\text{eff}}^{-1}$ for a square lattice of infinite *Si* bars, embedded in Al, versus the *Si* filling-fraction *f*. The bars have a rectangular cross section, whose sides are parallel to the *x*- and *y*-axes and in a ratio of 1:2, respectively [**35**], see Fig. 1(b). Each principal axis of *Si* and Al cubic crystals has been oriented parallel to a principal axis of the two-dimensional (2D) PC. The calculations were carried out with a lattice constant $a = 0.01$m in the limit $\omega \to 0$ $(k \to 0)$. In this case, the effective mass-density tensor is diagonal: $\bar{\rho}_{\text{eff}} = \left(\rho_{\text{eff}}\,\delta_{ij}\right)$, where ρ_{eff} is a linear function of *f* (**see Fig. 2(a)**), i.e. $\rho_{\text{eff}}(f) = \rho_{\text{Al}}(1-f) + \rho_{\text{Si}}f$.

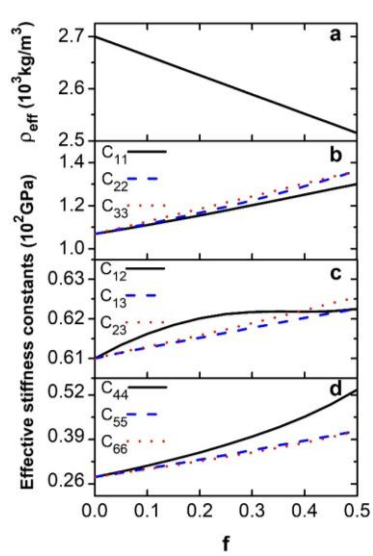


Fig. 2. (Color online). Graphs of the effective mass density (a) and elastic stiffness constants [(b), (c), and (d)] for a square lattice of *Si* rectangular bars embedded in *Al* versus the *Si* filling fraction *f*, which were calculated in the quasistatic limit($\omega \to 0, k \to 0$).

On the other hand, the effective stiffness tensor $\bar{C}_{eff}$ has nine independent elements in the interval $0 < f < 0.5$ just like an orthorhombic crystal (**Figs. 2(b), 2(c)**, and **2(d)**). At $f = 0$, the system has cubic symmetry since $\bar{C}_{eff} = \bar{C}_{Al}$. If $f = 0.5$, the *Si* bars touch each other with their faces perpendicular to the y-axis and the 2D PC transforms into a 1D PC. Hence, the homogenized PC acquires tetragonal symmetry for which there are only six independent stiffness constants (see **Fig. 2**). We have verified that our calculations in the quasistatic limit for *Si*/*Al* phononic crystals with either 1D or 2D periodicity coincide with the effective parameters predicted by the finite-element (FE) and asymptotic-homogenization (AH) methods [**36,37**].

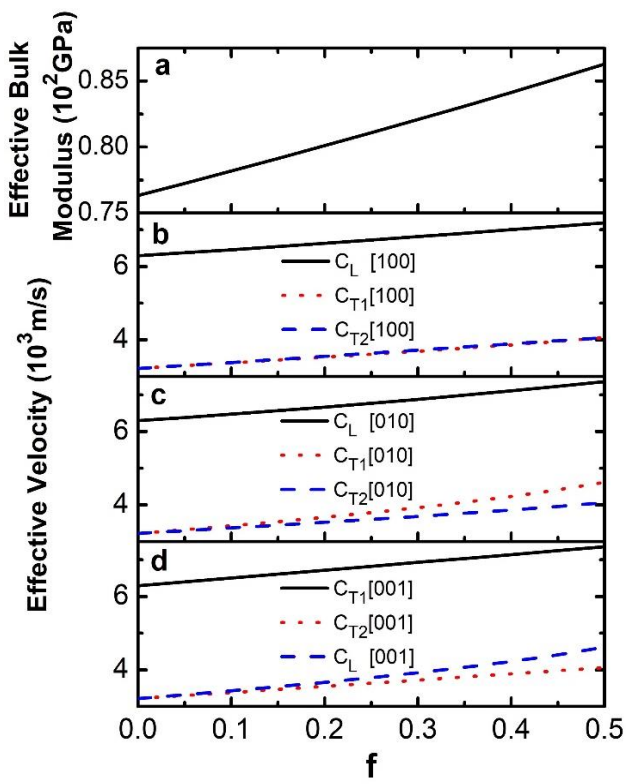


Fig. 3. (a) Effective bulk modulus and Effective sound velocities ((b) in the direction [100], (c) in the direction [010] and (d) in the direction [001]) for a square lattice of *Si* rectangular bars embedded in *Al* versus the *Si* filling fraction *f*.

It is very interesting to see that with the results shown in Fig. 2 and considering the well-known formulations of the acoustic physics of the orthorhombic crystalline symmetry (Eqs. (18)-(21)) [**38-41**], the bulk modulus and the velocities of sound in the periodic structure as a function of *f* are easily determined (see **Fig. 3**). Unlike other methods whose effective approximation of velocity is based on the expansion of plane waves of the velocity and elastic modulus fields present in the Christoffel equation [**29**].

Bulk modulus:

$$B = 1/9\,(C_{11} + C_{22} + C_{33} + 2C_{12} + 2C_{13} + 2C_{23}). \quad (18)$$

Velocity of sound:

(i) For waves propagating in the direction [100], $n_y = n_z = 0$ and $n_x = 1$, therefore:

$$v_x = c_L = \sqrt{\frac{C_{11}}{\rho}},\ v_y = c_{T1} = \sqrt{\frac{C_{66}}{\rho}} \text{ and } v_z = c_{T2} = \sqrt{\frac{C_{55}}{\rho}}. \quad (19)$$

(ii) For waves propagating in the direction [010], $n_x = n_z = 0$ and $n_y = 1$, therefore:

$$v_x = c_{T1} = \sqrt{\frac{C_{66}}{\rho}},\ v_y = c_L = \sqrt{\frac{C_{22}}{\rho}} \text{ and } v_z = c_{T2} = \sqrt{\frac{C_{44}}{\rho}}. \quad (20)$$

(iii) For waves propagating in the direction [001], $n_x = n_y = 0$ and $n_z = 1$, therefore:

$$v_x = c_{T1} = \sqrt{\frac{C_{55}}{\rho}},\ v_y = c_{T2} = \sqrt{\frac{C_{44}}{\rho}} \text{ and } v_z = c_L = \sqrt{\frac{C_{33}}{\rho}}. \quad (21)$$

where the subscripts T and L indicate transversal and longitudinal, respectively.

Formula (16) can also be applied for determining the effective parameters of PC with a liquid component having zero shear modulus ($\mu = 0$). This is possible by using a very small value of μ ($\mu \to 0$) in order for the stiffness matrix of the liquid ($\bar{C}_{\text{liq}}$) to be invertible. With this mathematical artifice, we could calculate the effective mass density and acoustic parameters for square and hexagonal lattices of infinite metallic cylinders (medium *a*) embedded in water (medium *b*) as those studied in Ref. [**30**] by means of multiple scattering theory. We found that if the cylinders are isolated, the effective stiffness constants $C_{\text{eff},44}$, $C_{\text{eff},55}$ and $C_{\text{eff},66}$ vanish with the auxiliary parameter $\mu \to 0$, whereas the constants $C_{\text{eff},ij}(i,j = 1,2,3)$ tend to the value of the effective bulk modulus B_{eff}. Our results for a square lattice of *Al* cylinders in water (see **Fig. 1(c)**), obtained with $ka \to 0$ and a frequency ω, satisfying $0 < \sqrt{\mu/\rho_b} < (\omega a/2\pi) << \sqrt{B_b/\rho_b}$, are shown in **Fig. 4**. The effective mass-density matrix $\bar{\rho}_{\text{eff}}$ turns out to be diagonal with principal values: $\rho_{\text{eff},xx} = \rho_{\text{eff},yy}$ and $\rho_{\text{eff},zz}$ (the cylinders are parallel to the *z*-axis). Their dependence on the cylinders radius *R* quantitatively agrees with the results of the work [**37**] (Fig. 10 therein). Notice that the calculated effective bulk modulus B_{eff} coincides with the predictions of the well-known formula [**16,28,42**]:$(1/B_{\text{eff}}) = (1/B_a)f + (1/B_b)(1-f)$. Consequently, a good agreement between our result for the effective (longitudinal) sound velocity $c_{\text{eff}} = \sqrt{B_{\text{eff}}/\rho_{\text{eff},xx}}$ and that obtained in Ref. [**30**] is also observed (panel (**c**) in **Fig. 4**).

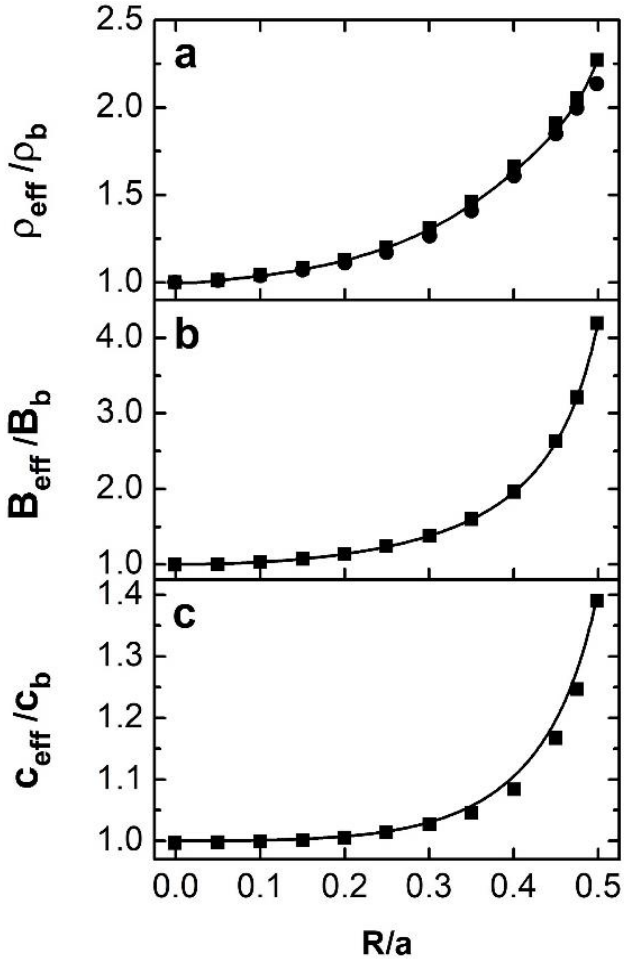


Fig. 4. Calculated dependences of the effective mass density ($\rho_{\mathrm{eff},xx} = \rho_{\mathrm{eff},yy}$ (squares) and $\rho_{\mathrm{eff},zz}$ (cyrcles), panel (a)), bulk modulus (b) and longitudinal sound velocity (c) for a 2D square array of Al cylinders embedded in water on the cylinder radius *R*. Solid lines were calculated in Ref. [**29**].

To illustrate the nonlocal effects, derived from the wave vector dependence of the effective matrix $\bar{\bar{A}}_{\mathrm{eff}}(\omega,\vec{k})$ (16), we have calculated the effective dynamic parameters for a rubber/Al 1D PC (**Figs. 5** and **6**) [**43**], see **Fig. 1(a)**. The thicknesses of rubber and *Al* slabs are $d_{\mathrm{rubber}} = 0.1a$ and $d_{Al} = 0.9a$ with $a = 0.01m$. In particular, we have considered transverse phonon modes propagating along the growth direction of the 1D PC, whose dispersion relation ($\vec{k}_0(\omega) = k_0(\omega)\hat{z}$) can be analytically described [**44**]. In panels (**b**) and (**c**) of **Fig. 5** and **Fig. 6**, we compare the frequency dependences of the effective mass density $\rho_{\mathrm{eff},yy}(\omega,\vec{k})$ and stiffness constant $C_{44,\mathrm{eff}}(\omega,\vec{k})$ computed in both *local* ($ka \to 0$) and *nonlocal* ($\vec{k} \to \vec{k}_0(\omega)$) regimes. **Figs. 5(a)** and **6(a)** show that the dispersion relation for transverse modes, propagating in the homogenized PC, i.e.

$$k_z(\omega) = \omega \lim_{\vec{k}\to\vec{k}_0(\omega)} \sqrt{\frac{\rho_{\mathrm{eff},yy}(\omega,\vec{k})}{C_{44,\mathrm{eff}}(\omega,\vec{k})}}, \qquad (22)$$

and the exact phononic dispersion (solid line therein) are identical. The local effective parameters (see dashed lines) reproduce the phononic dispersion only near the center of the first BZ. It is very interesting that the homogenized PC behaves as a double-negative elastic metamaterial in the frequency interval, corresponding to the second propagating band (**Fig. 6**). Indeed, in such band the effective dynamic mass density $\rho_{\mathrm{eff},yy}$ and the stiffness constant $C_{44,\mathrm{eff}}$ are both negative. Moreover, the effective mass density as well as k_z vanish at the top of the band. We should comment that the sign of k_z in the pass bands was determined by introducing a small dissipative part [**45,46**] in the shear modulus and imposing that $\Im(k_z)$ be positive. As a result, $\Re(k_z)$ ($|\Re(k_z)| \gg \Im(k_z)$) is positive (negative) in the first (second) phononic band. In the band gap, k_z and the nonlocal effective elastic parameters turn out to have a noticeably-large imaginary part.

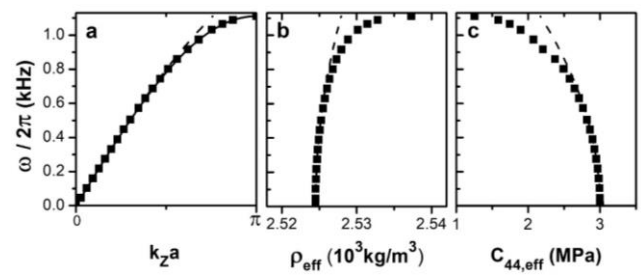


Fig. 5. (a) Lowest frequency band for transverse modes propagating in a 1D rubber/Al PC. Here, squares (dashed line) were (was) obtained by using the nonlocal (local) effective mass density $\rho_{\mathrm{eff}} = \rho_{\mathrm{eff},yy}$ and stiffness constant $C_{44,\mathrm{eff}}$, which are respectively shown in panels (b) and (c). Solid line in (a) corresponds to the exact analytical phononic dispersion.

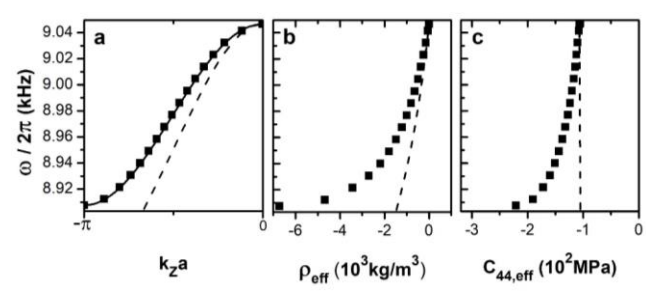


Fig. 6. The same as in **Fig. 5**, but for the second propagating band of transverse modes.

Conclusion & future prospective

Using a general homogenization theory, based on the Fourier formalism, we have calculated in the low-frequency limit the effective dynamic mass density and compliance tensors for 1D and 2D phononic crystals, for the latter case with constituents solid-solid and solid-fluid. Also, we have calculated the effective sound velocities and the nonlocal effective parameters, namely mass density and stiffness tensor, for a 1D solid phononic crystal. The calculated effective parameters allow us to describe the phononic band structure of the phononic crystal not only in the local low-frequency limit, but also beyond it. Besides, we have shown

that the anisotropy in the effective dynamic mass-density, appearing at sufficiently large frequencies. Our results illustrate the usefulness of the applied nonlocal homogenization approach, based on the Fourier formalism. However, such an approach still requires the calculation of sums over a large number of reciprocal lattice vectors for a good accuracy of the results, especially when there is a high contrast of the mass density and elastic moduli of the materials in the unit cell of the phononic crystal. It should be noted that the iterative procedure, applied here to determine the phononic dispersion relation $\vec{k}(\omega)$, provides only one solution, which depends on the chosen initial value for the wave vector. Unlike others homogenization approaches, our theory is rather general since it provides explicit expressions for both the effective local and nonlocal mass density and compliance tensors of arbitrary phononic crystals.

In conclusion, we have derived explicit expressions for the tensors of the effective nonlocal elastic response of arbitrary phononic crystals. As seen in the previous sections, this theory provides the basis for creating a periodic metamaterial with homogenized properties, such as anisotropy in its physical properties, negative density and negative elastic moduli. Due to the generality of our results, they will be useful to design elastic or acoustic metamaterials with low losses in wide frequency ranges. Today our work-team has extensive advances based on this present approach for three-dimensional phononic crystals, which will be reported shortly.

Acknowledgements

Authors thank J.A. Otero, J. Hernández-Cabanas, and G. Monsiváis-Galindo for their stimulating commentaries on the present work, and for calculating the effective parameters of *Si*/*Al* PCs using AH and FE methods to compare with our predictions in the quasistatic limit.

Conflicts of interest
There are no conflicts to declare.

Keywords

Metamaterial, phononic crystal, homogenization, effective parameters.

Received: 23 May 2020
Revised: 27 July 2020
Accepted: 01 September 2020

References

1. Pendry, J. B.; Holden, A. J; Stewart, W. J.; Youngs, I.; *Phys. Rev. Lett.*, **1996**, *76*, 4773.
2. Pendry, J. B.; Holden, A. J.; Robbins, D. J.; Stewart, W. J.; *J. Phys. Condens. Matter*, **1998**, *10*, 4785.
3. Smith, D. R.; Vier, D. C.; Padilla, W. J.; Nemat-Nasser, S. C.; Schultz, S.; *Appl. Phys. Lett.*, **1999**, *75*, 1425.
4. Pendry, J. B.; Holden, A. J.; Robbins, D. J.; Stewart, W. J.; *IEEE Trans. Microwave Theory Tech.*, **1999**, *47*, 2075.
5. Smith, D. R.; Padilla, W. J.; Vier, D. C.; Nemat-Nasser, S. C.; Schultz, S.; *Phys. Rev. Lett.*, **2000**, *84*, 4184.
6. Pendry, J. B.; *Phys. Rev. Lett.*, **2000**, *85*, 3966.
7. Veselago, V.G.; *Soviet Physics Uspekhi*, **1968**, *10*, 509.
8. Ramakrishna, S.; Grzegorczyk, T. (Ed. 1st); Physics and Applications of Negative Refractive Index Materials; CRC Press, USA, **2008**.
9. Walser, R. M.; Metamaterials: An introduction; Weiglhofer, Werner S.; Lakhtakia, Akhlesh (Eds.); SPIE-Digital Library, **2003**, pp. 1-22.
10. Shelby, R.A; Smith, D.R; Schultz S; *Science*, **2001**, 292, 5514.
11. Schuring, D.; Mock, J.J; Justice, B.J; Cummer, S.A; Pendry, J.B; Smith, D.R.; *Science*, **2006**, *314*, 5801.
12. Cerdán-Ramírez, V.; Zenteno-Mateo, B.; Sampedro, M. P.; Palomino-Ovando, M. A.; Flores-Desirena, B.; Pérez-Rodríguez, F.; *J. Appl. Phys.*, **2009**, *106*, 103520.
13. Reyes-Avendaño, J. A.; Alfredo-Badillo, U.; Halevi, P.; Pérez-Rodríguez, F.; *New J. Phys.*, **2011**, *13*, 073041.
14. Li, J.; Chan, C.T.; *Phys. Rev.*, **2004**, *E70*, 055602(R).
15. Berryman, J. G.; *J. Acoust. Soc. Am.*, **1980**, *68*, 1809.
16. Mei, J.; Liu, Z.; Wen, W.; Sheng, P.; Phys; *Rev. Lett.*, **2006**, *96*, 024301.
17. Torrent, D.; Håkansson, A.; Cervera, F.; Sánchez-Dehesa, J.; *Phys. Rev. Lett.*, **2006**, *96*, 204302.
18. Torrent, D.; Sánchez-Dehesa, J.; *Phys. Rev. B.*, **2006**, *74*, 224305.
19. Fokin, V.; Ambati, M.; Sung, C.; Zhang, X.; *Phys. Rev. B.*, **2007**, *76*, 144302.
20. Smith, D. R.; Schultz, S.; Markos, P.; Soukoulis, C. M.; *Phys. Rev. B.*, **2002**, *65*, 195104.
21. Chen, X.; Grzegorczyk, T. M.; Wu, B. I.; Pacheco, J.; Kong, J. A.; *Phys. Rev.*, **2004**, *E.70*, 016608.
22. Fink, M.; *Nat. Mater.*, **2014**, *13(9)*, 848-849.
23. Wu, Y.; Lai, Y.; Zhao-Qing, Z.; *Phys. Rev. B.*, **2007**, *76*, 205313.
24. Wu, Y.; Lai, Y.; Zhao-Qing, Z.; *Phys. Rev. Lett.*, **2011**, *107*, 105506.
25. Gutiérrez-Reyes, E.; Flores-Méndez, J.; González, A. L.; Pérez-Rodríguez, F.; *Proc. SPIE*, **2015**, *9438*, 94381H.
26. Reyes-Avendaño, J. A.; Sampedro, M. P.; Juárez-Ruiz, E.; Pérez-Rodríguez, F.; *J. Opt.*, **2014**, *16*, 065102.
27. Norris, A. N.; Shuvalov, A. L.; Kutsenko, A. A.; Proc. R. Soc. London, Ser., **2012**, *A 468*, 1629.
28. Krokhin, A. A.; Arriaga, J.A.; Gumen, L.N.; *Phys. Rev. Lett.*, **2003**, *91*, 264302.
29. Flores-Méndez, J.; Salazar Villanueva, M.; Ambrosio Lázaro, R. C.; Calixto Sirene, B.; Mota González, M. L.; Candia García, F.; *Advances in Materials Science and Engineering*, **2016**, *3017835*, 1-9.
30. Khelif, A.; Adibi, A. (Eds.), Phononic Crystals: Fundamentals and Applications; Springer (Eds.); Springer, NY, USA, **2016**.
31. Jin, Y.; Djafari-Rouhani, B.; Torrent, D.; *Nanophotonics*, **2019**, *8*, 685.
32. Flores-Méndez, J.; Pérez Rodríguez, F.; *Europhysics Letters*, **2013**, *103*, 54001.
33. Flores-Méndez, J.; Salazar Villanueva, M.; Hernández-Rodríguez, S.; Rodríguez Mora, J. I.; *Results in Physics*., **2017**, *7*, 1376.
34. Ortiz, G. P.; Martínez-Zérega, B. E.; Mendoza, B. S.; Mochán, W. L.; *Phys. Rev. B.*, **2009**, *79*, 245132.
35. The parameters used for Al (Si) are ρ = 2700 (2330) kg/m^3, C_{11}= 107.3 (166.2) GPA, C_{12}= 60.9 (64.4) GPA, C_{44}= 28.3 (79.8) GPA.
36. Berger, H.; Gabbert, U.; Köppe, H.; Rodriguez-Ramos, R.; Bravo-Castillero, J.; Guinovart-Diaz, R.; Otero, J. A.; Maugin, G. A.; *Comput. Mech.*, **2003**, *33*, 61.
37. Bravo-Castillero, J.; Rodríguez-Ramos, R.; Mechkour, H.; Otero, J. A.; Hernández-Cabanas, J.; Maykel Sixto, L.; Guinovart-Díaz, R.; Sabina, F. J.; *J. Mech. Mater. Struct.*, **2009**, *4*, 819.
38. Charles, K. (Ed. 5st.); Introduction to Solid State Physics; John Wiley & Sons, Inc., NY, USA, **1995**.
39. Ashcroft, N. W.; Mermin, N. D. (Ed. 1st); Solid State Physics; Saunders College Publishing, FL, USA, **1976**.
40. Lempriere, B. M. (Ed. 1st); Ultrasound and Elastic Waves; Academic Press, NY, USA, **2002**.
41. Every, A. G.; *Phys. Rev. B.*, **1980**, *22*, 4.
42. Torrent, D.; Sánchez-Dehesa, J.; *New J. Phys.*, **2008**, *10*, 023004.
43. The rubber parameters used are: ρ = 945 kg/m^3, C_{11}=511.5 MPa, C_{44}= 0.3 MPa, $C_{12} = C_{11} - 2C_{44}$.
44. EI Boudouti, E. H.; Djafari-Rouhani, B.; Khourdifi, E. M.; Dobrzynski, L.; *Phys. Rev. B.*, **1993**, *48*, 10987.
45. Brekhovskikh, L. M.; Godin, O. A. (Ed. 1st); Acoustics of Layered Media I: Plane and Quasi-Plane Waves; Springer, Berlin, GER, **1998**.
46. Kushwaha, M. S.; Halevi, P.; Dobrzynski, L.; Djafari-Rouhani, B.; *Phys. Rev. Lett.*, **1993**, *71*, 2022.

Authors biography

J. Flores Méndez. He is a Doctor in Materials Science. In addition, he has a Master of Science in Mechanical Engineering. Currently, he is a Professor-Researcher at the Faculty of Engineering of the Benemérita Universidad Autónoma de Puebla and at the Technological University of Puebla, Mexico. He is part of the National System of Researchers of Mexico. His research area focuses on studying the acoustic and electromagnetic response in metamaterials.

R. Pérez Cuapio. He received the degree of Master and Doctor in Semiconductor Devices. Currently, he is a Research Professor at the Faculty of Chemical Engineering of the Benemérita Universidad Autónoma de Puebla, Mexico. He is part of the National System of Researchers of Mexico. His line of research is the synthesis and characterization of nanomaterials applied in different areas, including: photocatalysis, solar cells and gas sensors.

C. Bueno Avendaño. He received the degree of Doctor in Semiconductor Devices. Currently, he is a Research Professor at the Faculty of Engineering of the Benemérita Universidad Autónoma de Puebla and at the Technological of Apizaco, Mexico. He is part of the National System of Researchers of Mexico. His research area is Nanoscience in nano and microstructures of ZnO and TiO_2, gas sensors, photo sensors and metal oxide solar cells.

Martín Hernández-Ordoñez. He received the B.E. degree in electronics engineering, and the M.Sc. and PhD degree in electrical engineering from the Universidad Autónoma de San Luis Potosí, Mexico in 2004 and 2007, respectively. He is currently working in the Veracruz Technological Institute, TecNM. His main research interests are in the fields of mathematical modeling and control of biological and fuzzy systems, robust and geometric control, and development of educational interfaces and robotics.

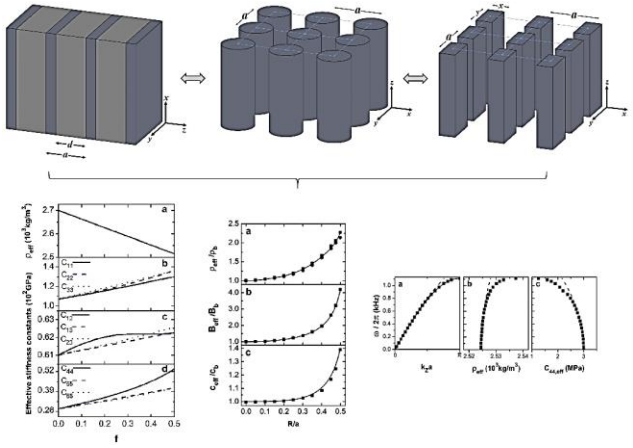
M. Aparicio Razo. He holds the title of Mechanical Engineer and the degree of Master in Automation from the National Technological Institute of Mexico and the Benemérita Universidad Autónoma de Puebla, respectively. He is a Full Time Research Professor at the Faculty of Electronic Sciences of the Benemérita Universidad Autónoma de Puebla, México. His research area focuses on studying new mechanical and electromagnetic properties in materials for industrial applications.

F. Candía García. He is a Mechanical Engineer, holds a Master's Degree in Structural Analysis, Master's and Doctorate in Education Sciences. He is a Professor-Researcher at the Faculty of Engineering of the Benemérita Universidad Autónoma de Puebla, México. He is part of the National System of Researchers of Mexico. His research area focuses on the study of new mechanical, electrical and electromagnetic properties in materials for industrial applications.

I. Vivaldo De La Cruz. He received the Master's and Doctorate degree in semiconductor devices. Subsequently, he carried out a research stay in Applied Electronics at the National Institute of Astrophysics, Optics and Electronics. He is currently a Professor-Researcher at the Faculty of Electronic Sciences of the Benemérita Universidad Autónoma de Puebla, Mexico. He is part of the National System of Researchers of Mexico. His research area is the synthesis and characterization of new semiconductor materials.

R. C. Ambrosio Lázaro. He is a Doctor of Electronic Sciences. Previously, he studied the Master of Electronic Sciences and Electronic Engineering. He is currently a Professor-Researcher at the Faculty of Electronic Sciences of the Benemérita Universidad Autónoma de Puebla, Mexico. He is part of the National System of Researchers of Mexico. His research area is the synthesis and characterization of new semiconductor materials.

B. Zenteno Mateo. He received the Doctor degree in applied physics from the Benemérita Universidad Autónoma de Puebla, Mexico. He is a Full Time Professor-Researcher at the Faculty of engineering. He is part of the National System of Researchers of Mexico. His research interests are in the areas of artificial EM materials and metamaterials.

Graphical abstract

Numerical Simulations present an approach for efficient, accurate calculations of the elastic wave properties of phononic crystals. We have derived formulas that along their application perfectly describe the effective non-local response in 1D elastic metamaterials, in the long-wavelength limit the propagation of elastic waves in 1D and 2D solid-solid media, as well as, solid-liquid media.